Backpaper - Computer Science 2 (2020-21)
Time: 3 hours.
Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. How is the number -50.875 stored as a floating point number in the computer ? Give the sign, mantissa and exponent. **[5 marks]**

2. How many floating point numbers are there in $[1, 4]$? **[4 marks]**

3. Let $g : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ be the function $g(x) = \frac{1}{1+x}$. Consider the sequence $x_{n+1} = g(x_n)$, starting with $x_0 = \frac{1}{2}$.
   (a) Show that the sequence x_n converges. **[3 marks]**
   (b) What does it converge to? **[3 marks]**

4. Consider solving the equation $f(x) = 0$.
   (a) Describe Newton's method, Secant method and Bisection method. **[5 marks]**
   (b) Explain using graphs the evolution of the iterates in the Newton method and the Secant method. **[4 marks]**

5. Use Gaussian elimination to solve
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 6 \end{bmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
[5 marks]

6. Use Gram-Schmidt orthogonalization process to find an orthonormal basis for the span of the following vectors in $\mathbf{R}^4$:
$$\Big\{(1, 2, 1, 1),\ (1, 0, 2, 0),\ (1, 3, 1, 1)\Big\}.$$
[5 marks]

7. Consider the real symmetric matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$
Implement *only* the first step of the classical Jacobi method to find the eigenvalues of the matrix, i.e. find the matrix $\mathbf{A}^{(1)}$ obtained from $\mathbf{A}^{(0)} = \mathbf{A}$. **[5 marks]**

8. Consider an infinitely differentiable function $f : [a, b] \to \mathbf{R}$.
   (a) Write down the Newton-Cotes formula for $\int_a^b f(x)dx$ with 4 equally spaced points $a = x_0 < x_1 < x_2 < x_3 = b$. **[4 marks]**
   (b) What is the *order* of the error in approximating the integral by the approximation? **[2 marks]**

9. Consider a function $z : [0, 1] \to \mathbf{R}$ with $z(0) = 1$, and which solves
$$\frac{dz(t)}{dt} = f\left(t, z(t)\right), \quad 0 \le t \le 1.$$
Assume that the function $f(t, x)$ is continuously differentiable with
$$\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial t}\right| \le M.$$
Now consider the sequence x_0, x_1, x_2, given by $x_0 = 1$ and
$$x_{i+1} = x_i + \frac{1}{2} \cdot f\left(\frac{i}{2}, x_i\right), \quad i = 0, 1.$$
Give a bound on $|z(1) - x_2|$. **[5 marks]**